1. *Answer true or false. No justification needed.*

   (a) The set $S = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c \in \mathbb{R},\ d$ is an integer$\}$ is a vector space with usual addition and scalar multiplication.

   Solution: False. □

   (b) The set $T = \{p(x) | p(x)$ is a polynomial with real coefficients and $p(1) = 0\}$ is a $\mathbb{R}$-vector space with usual multiplication and scalar multiplication.

   Solution: True. □

   (c) Every homogeneous system of 3 equations and 5 unknowns over $\mathbb{R}$ have infinitely many solutions.

   Solution: True. □

   (d) Every homogeneous system of 5 equations and 3 unknowns over $\mathbb{R}$ have only one solution.

   Solution: False. □

   Let $M(m, n)$ denote the F-vector space of $m \times n$ matrices with entries in F. Let $T : M(3, 2) \oplus M(2, 3) \to M(3, 3)$ be the function $T(A, B) = AB$.

   (e) T is a linear map.

   Solution: False. □

   (f) T is onto.

   Solution: False. □

2. Show that $V = \{(a, b, c, d) : a, b, c, d \in \mathbb{R}, a - b + 2c = 0\}$ is a subspace of $\mathbb{R}^4$. Compute its dimensions and find a basis of V.

   Solution: Let $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V$. Then let us consider the element

   $$\alpha.(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) \in \mathbb{R}^4 \text{ for any } \alpha \in \mathbb{R}.$$

   Since $a_1 - b_1 + 2c_1 = 0, a_2 - b_2 + 2c_2 = 0$, it immediately follows that

   $$(\alpha.a_1 + a_2) - (\alpha.b_1 + b_2) + 2(\alpha.c_1 + c_2) = 0.$$

Therefore,

$$\alpha.(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) \in V \text{ for any } \alpha \in \mathbb{R}.$$

i.e., V is a subspace of $\mathbb{R}^4$.

Since, $b = a + 2c$ for any $(a, b, c, d) \in V$, dimension of V is 3 and a basis can be given as follows:

$$\{(0, 0, 0, 1), (1, 1, 0, 0), (0, 2, 1, 0)\} \subset V$$

It is easy to see that the set is linearly independent and dimension of V=3 implies that the above set forms a basis of V.

3. Let $\mathbb{P}_n$ be the vector space of all polynomials with real coefficients of degree at most n. Show that $B = \{x(x-1)(x-2)(x-3), x(x-1)(x-2)(x-4), x(x-1)(x-3)(x-4), x(x-2)(x-3)(x-4), (x-1)(x-2)(x-3)(x-4)\}$ is a basis of $\mathbb{P}_4$. Treating B as an ordered basis of $\mathbb{P}_4$, compute the coordinates of x with respect to B.

   Solution: Since number of elements in B is equal to the dimension of $\mathbb{P}_4$, it is enough to show that B is a linearly independent set.

   Let $f_1(x) = x(x-1)(x-2)(x-3),\ f_2(x) = x(x-1)(x-2)(x-4),\ f_3(x) = x(x-1)(x-3)(x-4),$ $f_4(x) = x(x-2)(x-3)(x-4),\ f_5(x) = (x-1)(x-2)(x-3)(x-4)$ Also, let us assume that

$$\sum_{i=1}^{5} a_i.f_i(x) = 0 \text{ for } a_i \in \mathbb{R}.$$

   For $x = 0$, $a_5.f_5(0) = 0$, i.e. $a_5 = 0$. Similarly for $x = 1, 2, 3, 4$, it follows that $a_i = 0$ for $i = 1, 2, 3, 4$, respectively.

   Hence, B is a basis of $\mathbb{P}_4$. Next, let us assume that

$$x = \sum_{i=1}^{5} x_i.f_i(x) \text{ for } x_i \in \mathbb{R}.$$

   Then,
   for $x = 0$, we get $x_5.f_5(0) = 0$, i.e. $x_5 = 0$.
   For $x = 1$, we get $1 = x_4.f_4(1)$, i.e. $x_4 = -1/6$.
   For $x = 2$, we get $2 = x_3.f_3(2)$, i.e. $x_3 = 1/2$.
   For $x = 3$, we get $3 = x_2.f_2(3)$, i.e. $x_2 = -1/2$.
   For $x = 4$, we get $4 = x_1.f_1(4)$, i.e. $x_4 = 1/6$.

   Therefore, coordinates of x with respect to the basis B are given as follows:

$$[x]_B = \begin{bmatrix} 1/6 \\ -1/2 \\ 1/2 \\ -1/6 \\ 0 \end{bmatrix}$$

   . $\square$

4. Find a reduced echelon form of the following matrix using elementary row operations.

$$\begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 2 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

Find all solutions to the system of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\ x_2 + x_3 + 2x_4 &= 1. \end{aligned}$$

Solution:

$R_2 \to R_2 - 2.R_1$,

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$,

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & -2 & 1 & 1 & 2 \\ 0 & 0 & 3 & 5 & 4 \end{bmatrix}$$

$R_3 \to R_3/3$, $R_2 \to -R_2/2$,

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & -1/2 & -1/2 & -1 \\ 0 & 0 & 1 & 5/3 & 4/3 \end{bmatrix}$$

Thus, the system of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\ x_2 + x_3 + 2x_4 &= 1 \end{aligned}$$

is reduced to the system of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ x_2 - 1/2x_3 - 1/2x_4 &= -1 \\ x_3 + 5/3x_4 &= 4/3. \end{aligned}$$

Let $x_4 = k \in \mathbb{R}$, then $x_3 = \frac{1}{3}(4 - 5k)$, $x_2 = -\frac{1}{3}(k+1)$, $x_1 = \frac{1}{3}(7k - 5)$. □

5. Let $\mathbb{P}_n$ be the vector space of all polynomials with real coefficients of degree at most n. Show that $B = \{x^2 - 1, x - 1, x\}$ is a basis of $\mathbb{P}_2$. Show that there exists a linear function $T : \mathbb{P}_2 \to \mathbb{R}^2$ such that

$$T(x^2 - 1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; T(x - 1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } T(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find the matrix of T with respect to B and the basis $B' = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$.

Solution: Let us first find out the coordinates of the following elements in $\mathbb{P}_2$ in terms of the basis B'.

$$T(x^2-1) = \begin{bmatrix}1\\2\end{bmatrix} = -\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix}$$

$$T(x-1) = \begin{bmatrix}1\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$$

$$T(x) = \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix}$$

Thus, the matrix of T with respect to B and B' is given by

$$\begin{bmatrix}-1 & 0 & 1\\ 2 & 1 & 0\end{bmatrix}.$$

□

6. Let ϕ be an endomorphism of a vector space V over a field F. Define eigen value and eigen vector of ϕ. Let A and B be non zero square matrices and D be a diagonal matrix such that $AB = BD$. Show that A has an eigen vector.

   Solution: For a given endomorphism $\phi : V \to V$, a non zero vector $x \in F^n$ (assuming $dim_F(V) = n$) and a constant scalar $\lambda \in F$ are called an eigenvector and its eigenvalue, respectively, if $\phi(x) = \lambda x$.

   Let $A = (a_{ij})_{1\leq i,j\leq n}$ and $B = (b_{ij})_{1\leq i,j\leq n}$ are square matrices. Also let $D = (d_{ij})_{1\leq i,j\leq n}$ such that $d_{ij} = 0$ if $i \neq j$.

   Since B is a non zero matrix, there exists a non zero column, say $B_1 = \begin{bmatrix}b_{11}\\b_{21}\\b_{31}\end{bmatrix}$ (WLOG). Then, it is clear by matrix multiplication that

$$A\begin{bmatrix}b_{11}\\b_{21}\\b_{31}\end{bmatrix} = d_{11}.\begin{bmatrix}b_{11}\\b_{21}\\b_{31}\end{bmatrix}.$$

   Thus, $\begin{bmatrix}b_{11}\\b_{21}\\b_{31}\end{bmatrix}$ is an eigen vector of A with eigen value d_{11}.

□